Ch3: Systems of linear equations

→ Linear equation:

An equation is called linear if the powers of all variables in the equation are 1, and it does not involve any product of variables.

Examples:

$\sqrt{2}x + 3y - 2z = 3$

$x - 2y + 3z = 7$

↳ General form for linear equation

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$$

→ Where $x_1, x_2, \dots, x_n$ called variables (unknowns)

$a_1, a_2, \dots, a_n$ and b called coefficients

↳ Special parts

$2x + 3y = 0$ is called homogeneous linear equation

وذلك لان قيمة المعادلة
تساوي 0

→ Linear equation systeme:

A finit set of linear equations

$$\begin{cases} a_{11} x_1 + a_{12} x_2 + a_{1n} x_n = b_1 \\ a_{21} x_1 + a_{22} x_2 + a_{2n} x_n = b_2 \\ a_{m1} x_1 + a_{m2} x_2 + a_{mn} x_n = b_m \end{cases}$$

Exemple :

$$\begin{cases} 2x + 3y - z = 1 \\ x - 2y + 3z = 2 \\ x + 2y = 5 \end{cases}$$

Systeme of linear equation

one solution — no solution — ifinitely many solution

* if there is at least one solution of the systeme it is called consistent

* if the system has no solution it is called inconsistent

$$\begin{cases} x - 2y = 0 \\ 2x + y = 0 \\ x + y = 0 \end{cases} \Longrightarrow \text{homogenous linear system (always consistent)}$$

→ Elementary row operations

1. Multiply a row by a skalar (nonzero constant)
2. Switch rows
3. Add a mulitple of one equation to another

→ Augmented matrix of systeme

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ a_{31} & a_{32} & \dots & a_{3n} & b_3 \end{array}\right]$$

← يمكننا حل linear system باستخدام الـ matrix اذا تمكنا من

تحويل الـ Augmented matrix للشكل التالي عن طريق الـ e. row operations

$$\left[\begin{array}{ccc|c} 1 & a_{12} & a_{13} & b_1 \\ 0 & 1 & a_{23} & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right]$$

Triangular form

diagonal

حتى لو لم تساوي 1 لا توجد مشكلة لكن إذا كانت 1 يكون الحل أسرع.

Exl:

Solve the following systeme of linear equation

$$\begin{cases} x + 3y + 6z = 25 \\ 2x + 7y + 14z = 58 \\ 2y + 5z = 19 \end{cases}$$

Step 1
$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{array}\right]$$
Augmented matrix

Step 2:

$R_2 - 2R_1 \to R_2$
$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 5 & 19 \end{array}\right]$$

$R_3 - 2R_2 \to R_3$
$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$
Triangular Form

Step 3: $x + 3y + 6z = 25$
$y + 2z = 8$
$z = 3$

$\Rightarrow y + 2z = 8$
$y = 8 - 2(3) = 2$

$\Rightarrow x + 3y + 6z = 25$
$x = 25 - 3(2) - 6(3)$
$x = 1$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Remarks !!

1. $$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

في حال واجهنا هذا الشكل
أثناء محاولة الوصول إلى
الـ triangular form هنا يعني:
$\Rightarrow$ (linear systeme has infinitely many sol)

$x + 3y + 2z = 4$
$y - 3z = 2$

let $z = t \to y = 2 + 3t \to x = -2 - 11t$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 - 11t \\ 2 + 3t \\ t \end{pmatrix}$$

2. $\left(\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right)$ أما هنا الشكل فهو يعني: $\Rightarrow$ (linear system has no solution)

Ex2 :

For wich values of "a" will the following systeme has no solution? exactly one solution? infinitely many sol?

$$\begin{cases} x + 2y - 3z = 4 \\ 3x - y + 5z = 2 \\ 4x + y + (a^2 - 14)z = a + 2 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right)$$

$R_2 - 3R_1 \to R_2$

$R_3 - 4R_1 \to R_3$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{array}\right)$$

$R_3 - R_2 \to R_3$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right)$$

For $a^2 - 16 = 0$ and $a - 4 = 0 \Rightarrow$ many solution

$$\begin{cases} a^2 - 16 = 0 \\ a - 4 = 0 \end{cases} \Rightarrow \begin{cases} a^2 = 16 \\ a = 4 \end{cases} \Rightarrow \begin{cases} a = \pm 4 \\ a = 4 \end{cases} \Rightarrow a = 4$$

For $a^2 - 16 = 0$ and $a - 4 \neq 0 \Rightarrow$ no solution

$a = -4$

For $a^2 - 16 \neq$ and $a - 4 \neq 0 \Rightarrow$ one solution $a \neq \pm 4$

→ Row echelon Form (REF)

Augmented matrix is REF if:

1. Any rows with all zeros are placed at the bottom of the matrix.
2. The first nonzero nb in the row is a 1, and we call it leader 1
3. Every leading 1 is to the right of the one above it

entry: first nb in the row after 0

Example:
$$\left(\begin{array}{ccccc|c} \boxed{1} & 0 & 6 & 5 & 8 & 2 \\ 0 & 0 & \boxed{1} & 2 & 7 & 3 \\ 0 & 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

→ Reduced Row echelon form (RREF)

+4. All entries in a column above and below a leading 1 are zeros.

Ex:
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

N.B !!

zero matrices $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ are RREF

→ Pivot position

Ex: $\begin{pmatrix} \boxed{1} & 2 & 3 & | & 4 \\ 3 & \boxed{2} & 1 & | & 6 \\ 4 & 4 & 4 & | & 10 \end{pmatrix} \xrightarrow{R.E.F} \begin{pmatrix} \boxed{1} & 2 & 3 & | & 4 \\ 0 & \boxed{1} & 2 & | & 3/2 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$

Pivot columns: 1^{st} and 2^{nd} columns

Ex1: $A = \begin{pmatrix} 0 & -5 & -4 \\ 1 & 4 & 3 \\ 5 & 10 & 7 \end{pmatrix}$ Find the RREF of A

$R_1 \leftrightarrow R_2$ $\begin{pmatrix} 1 & 4 & 3 \\ 0 & -5 & -4 \\ 5 & 10 & 7 \end{pmatrix}$

$R_3 - 5R_1 \to R_3$ $\begin{pmatrix} 1 & 4 & 3 \\ 0 & -5 & -4 \\ 0 & -10 & -8 \end{pmatrix}$

$R_3 - 2R_2 \to R_3$ $\begin{pmatrix} 1 & 4 & 3 \\ 0 & -5 & -4 \\ 0 & 0 & 0 \end{pmatrix}$

$\frac{R_2}{-5}$ $\begin{pmatrix} 1 & 4 & 3 \\ 0 & 1 & -4/5 \\ 0 & 0 & 0 \end{pmatrix}$

$R_1 - 4R_2 \to R_1$ $\begin{pmatrix} 1 & 0 & -1/5 \\ 0 & 1 & -4/5 \\ 0 & 0 & 0 \end{pmatrix}$

Ex 2:

$$\begin{cases} 2x+4y-3z=-1 \\ 5x+10y-7z=-2 \\ 3x+6y+5z=9 \end{cases}$$

Solve the systeme

$$\left(\begin{array}{ccc|c} 2 & 4 & -3 & -1 \\ 5 & 10 & -7 & -2 \\ 3 & 6 & 5 & 9 \end{array}\right)$$

$R_2 - \frac{5}{2}R_1 \to R_2$

$R_3 - \frac{3}{2}R_1 \to R_3$

$$\left(\begin{array}{ccc|c} 2 & 4 & -3 & -1 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{19}{2} & \frac{21}{2} \end{array}\right)$$

$2R_2 \to R_2$

$$\left(\begin{array}{ccc|c} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \frac{19}{2} & \frac{21}{2} \end{array}\right)$$

$R_3 - \frac{19}{2}R_2 \to R_3$

$$\left(\begin{array}{ccc|c} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$0 \neq 1$

doesn't have a solution

Ex 3:

$$\begin{cases} 3x-y-5z=9 \\ y-10z=0 \\ -2x+y=-6 \end{cases}$$

$$\left(\begin{array}{ccc|c} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ -2 & 1 & 0 & -6 \end{array}\right)$$

$R_3 + \frac{2}{3}R_1 \to R_3$

$$\left(\begin{array}{ccc|c} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ 0 & \frac{1}{3} & -\frac{10}{3} & 0 \end{array}\right)$$

$R_3 - \frac{1}{3}R_2 \rightarrow R_3$ $\begin{pmatrix} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\frac{R_1}{3} \rightarrow R_1$ $\begin{pmatrix} 1 & -\frac{1}{3} & -\frac{5}{3} & 3 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$R_1 + \frac{1}{3}R_2 \rightarrow R_1$ $\left(\begin{array}{ccc|c} 1 & 0 & -5 & 3 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$z = t$

$y - 10t = 0 \Rightarrow y = 10t$

$x - 5t = 3 \Rightarrow x = 3 + 5t$

$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3+5t \\ 10t \\ t \end{pmatrix}$

Ex4:

$$\begin{cases} x + 2y + z + w = 3 \\ x + y - z + w = 1 \\ x + 3y - z + w = 5 \end{cases}$$

$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 3 & -1 & 1 & 5 \end{array}\right)$

$R_2 - R_1 \rightarrow R_2$

$R_3 - R_1 \rightarrow R_3$ $\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 2 \end{array}\right)$

$-R_2 \rightarrow R_2$ $\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \end{array}\right)$

$R_3 - R_2 \to R_3$
$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$R_1 - 2R_2 \to R_1$
$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$w = t$
$z = s$
$y = 2$
$x - z + w = -1 \Rightarrow x - s + t = -1 \Rightarrow x = -1 + s - t$

→ Rank of matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 5 & 9 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad r = 2$$

The rank is the nb of leading 1

Theoreme:
Let A be the $m \times (n+1)$ augmented matrix corresponding to a consistent systeme of equations in n variables and suppose A has rank r.

1. $r = n \Rightarrow$ unique solution
2. $r < n \Rightarrow$ infinitely many solution